Int. J. Multiphase Flow Vol. 7, No. 6, pp. 719–727, 1981
Printed in Great Britain.

0301-9322/81/060719–09$02.00/0
Pergamon/Elsevier

BRIEF COMMUNICATION

ON THE CALCULATION OF STOKES' FLOW PAST POROUS PARTICLES

J. J. L. HIGDON† and M. KOJIMA
Department of Chemical Engineering, Stanford University, Stanford, CA 94305, U.S.A.

(*Received* 20 *April* 1981)

I. INTRODUCTION

The flow of a viscous fluid past a porous particle has been studied by several investigators in widely varying fields. The areas of application range from the flow over a packed bed to the modeling of polymer molecules as porous particles. These problems differ from the classic problem of flow through a porous medium, because the bodies are of finite extent and are bordered by regions of pure fluid. To calculate the fluid velocity inside the porous body, we require an equation which accurately models the flow in the boundary region near the surface and is consistent with Darcy's law far from the surface in the interior of the body. A modification of Darcy's law which meets these requirements for many materials was proposed by Brinkman (1947) in the form:

$$\nabla \cdot \hat{\boldsymbol{\sigma}} \equiv -\nabla \hat{p} + \mu \nabla^2 \hat{\mathbf{u}} = \frac{\mu}{k}(\hat{\mathbf{u}} - \mathbf{u}^P), \qquad [1]$$

where $\hat{\mathbf{u}}$ is the volume average velocity, $\mathbf{u}^P$ the particle velocity, $\hat{p}$ the interstitial average pressure and $\hat{\boldsymbol{\sigma}}$ is defined by

$$\hat{\sigma}_{ij} \equiv -\hat{p}\delta_{ij} + \mu\left(\frac{\partial \hat{u}_i}{\partial x_j} + \frac{\partial \hat{u}_j}{\partial x_i}\right).$$

The permeability k is a physical constant determined by the porous material.

This equation is based on Stokes' equation for low Reynolds number flow with the addition of a drag force $\frac{\mu}{k}(\hat{\mathbf{u}} - \mathbf{u}^P)$ to account for the local resistance due to the presence of the particle. Debye & Bueche (1948) followed a similar line of reasoning and arrived at Brinkman's equation independently. Although [1] was originally derived from heuristic arguments, it has since received theoretical justification from numerous authors (see Howells 1974). The equation has been verified experimentally for some specific materials, but it has its limitations and cannot be used indiscriminately. A summary of the experimental work is included in the appendix to this paper together with a crude criterion for assessing the validity of the model for a given material.

An alternative approach to the subject is to employ Darcy's law, but to allow discontinuities in the fluid velocity at the surface of the body. This idea was suggested by Beavers & Joseph (1967) who proposed an expression for this "slip" boundary condition and found reasonable agreement with experiments. Saffman (1971) and others have offered theoretical confirmation for this approach. In practice, the Brinkman equation appears more useful, because it incorporates a more fundamental analysis and embodies a well-defined stress tensor.

†Present address: Department of Chemical Engineering, University of Illinois, Urbana, IL 61801, U.S.A.

Felderhof & Deutch (1975 a, b) and Felderhof (1975 a, b) used Brinkman's equation to model the hydrodynamic interaction of polymer molecules. In that series of papers, they derived [1], constructed a formal solution in terms of an integral equation and outlined methods for calculating physical properties of the polymers, such as the translational friction and the intrinsic viscosity for dilute suspensions of spherical molecules.

In the present note, we summarize some previous work on porous media and extend to this subject a few of the well-known results for flow past an impermeable particle—a problem which has been studied extensively in the past. Although these results are easily derived from basic principles, many have not been cited in the literature on porous media and have not been exploited in the analysis of problems in this area. Finally, we derive certain asymptotic results for small and large permeability which greatly simplify the analysis for the case of particles with arbitrary geometry.

II. STATEMENT OF THE PROBLEM

We consider the problem of a porous particle moving through a viscous fluid with velocity $\mathbf{u}^P = \mathbf{U}^P + \mathbf{\Omega}^P \times \mathbf{x}$ where $\mathbf{U}^P$ and $\mathbf{\Omega}^P$ are constants. The undisturbed velocity field is $\mathbf{u}^\infty$. We assume that the Reynolds number for the motion is small, and hence that the flow outside the particle is governed by Stokes' equation:

$$\nabla\cdot\boldsymbol{\sigma} \equiv -\nabla p + \mu\nabla^2\mathbf{u} = 0 \qquad [2]$$

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

while inside the particle, [1] applies. In addition, $\mathbf{u}$ is everywhere divergence free. For convenience, we have dropped the ˆ notation with the understanding the $\mathbf{u}$ and p refer to averaged quantities in the interior of the particle and their actual values exterior to the particle.

The boundary conditions for the equations are the continuity of velocity $\mathbf{u}$ and of the surface force $\mathbf{f} = \boldsymbol{\sigma}\cdot\mathbf{n}$ across the particle surface, where $\mathbf{n}$ is the unit normal to the surface. Finally, we require that $\mathbf{u}\to\mathbf{u}^\infty$ as $x\to\infty$.

This completes the statement of the problem.

III. INTEGRAL EQUATIONS

There are several ways in which solutions of the system of Eq. [1] and [2] may be calculated. One of the most convenient is to work with an integral equation for the velocity and the surface force. In following this approach, we use the appropriate Green's functions for [1] and [2] to construct integral solutions valid in their respective domains. The boundary condition for $\mathbf{u}$ and $\mathbf{f}$ then provides an integral equation extending over the surface of the particle.

The integral solution of [2] in the region exterior to the particle is well known. In particular, taking the limit as the point $\mathbf{x}$ approaches the surface, we have

$$\tfrac{1}{2}u_i(\mathbf{x}) = u_i^\infty(\mathbf{x}) + \int_{S^P} K_{0_{ijk}}(\mathbf{x}-\mathbf{y})u_j(\mathbf{y})n_K \mathrm{d}S_y - \int_{S^P} L_{0ij}(\mathbf{x}-\mathbf{y})f_j(\mathbf{y})\mathrm{d}Sy_y \qquad [3]$$

where

$$K_{0_{ijk}}(\mathbf{x}) = \frac{3}{4\pi}\frac{x_ix_jx_K}{r^5}, \quad L_{0_{ij}}(\mathbf{x}) = \frac{1}{8\pi\mu}\left(\frac{\delta_{ij}}{r} + \frac{x_ix_j}{r^3}\right). \qquad [4]$$

(See Ladyzhenskaya 1963, pp. 54–56).

In the interior of the particle, we employ the Green's functions for Brinkman's equation as found by Howells (1974). Proceeding as before, we take the limit as **x** approaches the surface and find:

$$\tfrac{1}{2}u_i(\mathbf{x}) = -\int_{S^P} K_{ijk}(\mathbf{x}-\mathbf{y})u_j(\mathbf{y})n_K \mathrm{d}S_y + \int_{S^P} L_{ij}(\mathbf{x}-\mathbf{y})f_j(\mathbf{y})\mathrm{d}S_y, \qquad [5]$$

where

$$K_{ijk}(\mathbf{x}) = \frac{\mathrm{e}^{-\alpha r}(1+\alpha r)}{4\pi}\left[\frac{\delta_{ik}x_j + \delta_{ij}x_k}{r^3} - \frac{2x_ix_jx_k}{r^5}\right] + \frac{1}{4\pi}\frac{x_i\delta_{jk}}{r^3}$$

$$+\frac{\mathrm{e}^{-\alpha r}(6+6\alpha r+2\alpha^2r^2)-6}{4\pi\alpha^2}\left[\frac{\delta_{ik}x_j+\delta_{ij}x_k+\delta_{jk}x_i}{r^5} - \frac{5x_ix_jx_k}{r^7}\right]$$

$$L_{ij}(\mathbf{x}) = \frac{\mathrm{e}^{-\alpha r}}{4\pi}\left[\frac{\delta_{ij}}{r} - \frac{x_ix_j}{r^3}\right] + \frac{\mathrm{e}^{-\alpha r}(1+\alpha r)-1}{4\pi\mu\alpha^2}\left[\frac{\delta_{ij}}{r^3} - \frac{3x_ix_j}{r^5}\right], \qquad [6]$$

and

$$\alpha = \frac{1}{\sqrt{k}}.$$

In view of the continuity of the velocity **u** and the surface force **f**, [3] and [5] constitute a system of integral equations for **u** and **f**. In principle, it is possible to obtain a numerical solution of these equations using a method analogous to that employed by Youngren & Acrivos (1975) for Stokes' flow past an impermeable particle; however, it will be more useful in some cases to consider asymptotic forms of the integral equations for $k \ll 1$ or $k \gg 1$.

Specifically, when the permeability is small with respect to a characteristic body dimension a (i.e. when $\frac{\sqrt{k}}{a} \ll 1$), the flow past the porous particle may be considered as a perturbation on the flow past an impermeable particle with the same geometry. Thus, to leading order, the velocity is zero on the surface, and the surface force **f** assumes the value **f***calculated for the impermeable body. By substituting these expressions into [5], we find the first order correction for **u**. To evaluate the integrals appearing in [5], we construct the inner expansion for **K** and **L** by defining an inner variable $\zeta = \alpha r$ and taking the limit as $\alpha \to \infty$ while holding ζ constant. With this approximation, the expression for $\mathbf{u}^{(1)}$ becomes:

$$u_j^{(1)}(\mathbf{x}) = \frac{\sqrt{k}}{\mu} f_k^*(\mathbf{x})[\delta_{jk} - n_jn_k] + O(k). \qquad [7]$$

Although the analysis leading to [7] is straightforward, the details are somewhat involved, and the result might be obtained more easily by considering an alternative method. For flow through a porous body, the velocity in a thin layer near the surface is determined primarily by the flow outside the body. The presence of this boundary layer allows a simplification of [1] analogous to that achieved in traditional boundary layer theory for high speed flow. In particular, we find that derivatives in the plane of the surface are of lower order than derivatives normal to the surface and may be neglected to leading order. The modified equation is easily solved in conjunction with the boundary conditions to yield an expression for $\mathbf{u}^{(1)}$ identical to [7].

Having found the first order term for the velocity $\mathbf{u}^{(1)}$, we can substitute for **u** in [3] to get the first order equation for **f**. This equation is identical to that for an impermeable particle immersed in a specified flow field which is determined by $\mathbf{u}^{(1)}$. Thus, we find that the integral equation for **f** at any order has the same form as the equation for **f***. For convenience, we define the integral operator $\mathscr{L}$ such that:

$$\mathscr{L}_{ij}f_j = \int_{S^P} L_{0ij}(\mathbf{x}-\mathbf{y})f_j(\mathbf{y})\mathrm{d}Sy. \qquad [8]$$

If we assume that this operator has been inverted as part of the solution for $\mathbf{f}^*$, then the solution for the first order term $\mathbf{f}^{(1)}$ follows immediately:

$$f_i^{(1)}(\mathbf{x}) = \frac{\sqrt{k}}{\mu}[\mathscr{L}^{-1}]_{ij}\left\{-\tfrac{1}{2}f_k^*(\delta_{jk} - n_j n_k) + \int_{S^P} K_{0_{jlm}} f_k^*(\delta_{kl} - n_k n_l) n_m \mathrm{d}S\right\}. \qquad [9]$$

The solution for $\mathbf{f}^{(1)}$ thus obtained may be used to find the next term in $\mathbf{u}$ by substituting in [5] and evaluating the resulting integrals. In principle, this iteration process may be continued to any order.

For the case of large permeability, it is also possible to construct an iterative solution, since in this limit, the solution is a perturbation on the undisturbed flow. Thus, to leading order $\mathbf{u} = \mathbf{u}^\infty$ and $\mathbf{f} = \mathbf{f}^\infty \equiv \boldsymbol{\sigma}^\infty \cdot \mathbf{n}$. Because the asymptotic limit as $\alpha \to 0$ (hence, $k \to \infty$) constitutes a regular perturbation, the functions $\mathbf{K}$ and $\mathbf{L}$ may be expanded in power series in α, and the appropriate equations solved at each order. We note that the leading order terms for $\mathbf{K}$ and $\mathbf{L}$ are $\mathbf{K}_0$ and $\mathbf{L}_0$ respectively. This result is obvious when we observe that Brinkman's equation reduces to Stokes' equation in the limit of large permeability.

In the expansions for $\mathbf{u}$ and $\mathbf{f}$, the odd powers of α can be shown to vanish identically, hence the series proceed in powers of α^2 (or k^{-1}). The first order corrections for $\mathbf{u}$ and $\mathbf{f}$ may be written:

$$u_i^{(1)}(\mathbf{x}) = \frac{1}{16\pi k}\int_{S^P}(u_j^\infty - u_j^P)n_k\left[-\frac{\delta_{ij}r_i + \delta_{ik}r_j + \delta_{ij}r_k}{r} + \frac{r_i r_j r_k}{r^3}\right]\mathrm{d}S$$
$$- \frac{1}{32\pi k\mu}\int_{S^P}\left[\frac{r_i r_j - 3r^2\delta_{ij}}{r}\right]f_j^\infty \mathrm{d}S, \qquad [10]$$

$$f_i^{(1)}(\mathbf{x}) = [\mathscr{L}^{-1}]_{ij}\left\{-\frac{1}{2}u_j^{(1)}(\mathbf{x}) + \int_{S^P} K_{0_{jlm}}(\mathbf{x}-\mathbf{y})u_l^{(1)}(\mathbf{y})n_m \mathrm{d}S y\right\}. \qquad [11]$$

In some cases for which the permeability is large, it may be more efficient to employ a different approach. By considering Brinkman's equation as an inhomogeneous form of Stokes' equation, we find an integral equation for $\mathbf{u}$ extending over the volume of the particle:

$$u_j(\mathbf{x}) = u_j^\infty(\mathbf{x}) - \frac{\mu}{k}\int_{V^P} L_{0_{jk}}(\mathbf{x}-\mathbf{y})[u_k(\mathbf{y}) - u_k^P(\mathbf{y})]\mathrm{d}V_y \qquad [12]$$

(see Felderhof & Deutch 1975 a). This equation is valid of course for all values of k, but it is most useful for large k.

For large permeability, we set $\mathbf{u} = \mathbf{u}^\infty$ as the initial approximation and iterate using [12]. This method avoids the difficulties of solving integral equations and requires only the evaluation of integrals. This resulting expansion for $\mathbf{u}$ has the form:

$$u_j = u_j^\infty - \frac{\mu}{k}\int_{V^P} L_{0_{jk}} u_k^\infty \mathrm{d}V + \frac{\mu^2}{k^2}\int_{V^P} L_{0_{jk}}\int_{V^P} L_{0_{kl}} u_l^\infty \mathrm{d}V\mathrm{d}V + O(k^{-3}). \qquad [13]$$

If the particle is held stationary in a uniform flow $\mathbf{U}^\infty$, the velocity to first order may be written:

$$u_j(\mathbf{x}) = U_j^\infty - \frac{U_k^\infty}{8\pi k}\int_{S^P}\left(\frac{-\delta_{jk}r_l + \delta_{kl}r_j}{r}\right)n_l \mathrm{d}S_y + O(k^{-2}), \qquad [14]$$

where $\mathbf{r} = \mathbf{x} - \mathbf{y}$.

We note that the method of solution based on the volume integral [12] may easily be generalized to include problems with non-uniform permeability.

Each of the techniques described in this section depends on numerical calculations for evaluating terms higher than first order. The details of individual problems will dictate whether a direct numerical solution would be more efficient than the methods suggested here.

IV. RECIPROCAL THEOREM

In the study of motion at low Reynolds number, it is possible to obtain some useful results without solving the equations of motion explicitly. One of these is a reciprocal theorem which relates the force, torque and stress due to a particle in a specified flow field to the same quantities in a different flow field. The basic form of the reciprocal theorem for Stokes' flow about impermeable particles is:

$$\int_{S^P} \mathbf{f}\cdot\mathbf{u}'\mathrm{d}S = \int_{S^P} \mathbf{f}'\cdot\mathbf{u}\mathrm{d}S \qquad [15]$$

where $\mathbf{u}$ and $\mathbf{f}$ are the velocity and surface force in one flow field and $\mathbf{u}'$ and $\mathbf{f}'$ refer to the same quantities in the second flow field. From this identity, the reciprocal relationships for the force, torque and stress are readily obtained. (See Hinch 1972.)

For porous particles, Reuland *et al.* (1978) showed that [15] applies as before and derived reciprocal relationships for the force and the torque on porous particles. Their derivation was straightforward, employing the divergence theorem to convert the surface integrals to integrals over the volume of the particle and substituting for $\nabla\cdot\boldsymbol{\sigma}$ from [1]. This derivation required that the permeability in the two particles be identical—a condition we find somewhat restrictive. In fact, it is a trivial modification to take the volume of integration outside the particle and show that [15] is independent of particle structure. Thus, the reciprocal theorem may be applied to particles of identical shape but different composition (e.g. one permeable and the other impermeable).

V. HYDRODYNAMIC FORCE

In this section, we turn our attention to the calculation of the hydrodynamic force acting on a porous particle immersed in a specified flow field. The force may be calculated by integrating the force vector $\mathbf{f}$ over the surface or by integrating $\frac{\mu}{k}(\mathbf{u}-\mathbf{u}^P)$ over the volume of the particle. The expressions for $\mathbf{u}$ and $\mathbf{f}$ required to evaluate these integrals may be obtained from the integral equations by one of the methods described in the previous section. Here we give the results for the hydrodynamic force in the asymptotic limits of large and small permeability.

In the case of large permeability, we have two techniques available—one based on the surface integral equation and the other based on the volume integral equation. Using each technique separately, we obtain two equivalent expressions for the force. The leading terms in each case are:

$$F_i = \int_{S^P} [\mathscr{L}^{-1}]_{ij}\left\{-\frac{1}{2}u_j^{(1)}(\mathbf{x}) + \int_{S^P} K_{0_{jlm}}(\mathbf{x}-\mathbf{y})u_l^{(1)}(\mathbf{y})n_m\mathrm{d}S_y\right\}\mathrm{d}S_x \qquad [16]$$

where $\mathbf{u}^{(1)}$ is given by [10], and

$$F_i = \frac{\mu}{k}\int_{V^P}[u_i^\infty - u_i^P]\mathrm{d}V_y - \frac{\mu^2}{k^2}\int_{V^P}\int_{V^P} L_{0ij}(\mathbf{x}-\mathbf{y})[u_j^\infty - u_j^P]\mathrm{d}V_x\mathrm{d}V_y. \qquad [17]$$

If the imposed velocity field $\mathbf{u}^\infty = \mathbf{U}^\infty$ is uniform and $\mathbf{u}^P = \mathbf{U}^P$ with $\boldsymbol{\Omega}^P = 0$, [17] may be written in the alternative form:

$$F_i = \left(U_j^\infty - U_j^P\right)\left\{\frac{\mu}{k} V^P\delta_{ij} - \frac{\mu}{8\pi k^2}\int_{S^P}\int_{S^P}(-\delta_{ij}n_l(\mathbf{x})n_l(\mathbf{y}) + n_i(\mathbf{x})n_j(\mathbf{y}))r\mathrm{d}S_x\mathrm{d}S_y + 0(k^{-3})\right\} \quad [18]$$

where $r = |\mathbf{x} - \mathbf{y}|$ and V^P is the volume of the particle.

As an example, we find that for a porous sphere of radius a, in a uniform flow $\mathbf{U}^\infty$, these expressions may be evaluated to yield

$$\mathbf{F} = \frac{4}{3}\pi a\mu\mathbf{U}^\infty\left[\frac{a^2}{k} - \frac{4}{15}\frac{a^4}{k^2} + O\left(\frac{a^6}{k^3}\right)\right].$$

For the case of low permeability, we observe that the iterative solution of the integral equation provides an expression for $\mathbf{u}$ to a given order, before calculating the value of $\mathbf{f}$ to that order. Since the specific form of $\mathbf{f}$ is not required, we employ the reciprocal theorem to find the total force in terms of $\mathbf{u}$ and the value of $\mathbf{f}$ at the previous order.

Consider two stationary particles of identical geometry—one impermeable, the other porous—immersed in identical uniform velocity fields $\mathbf{U}^\infty$. From the reciprocal theorem we have

$$\mathbf{U}^\infty\cdot\mathbf{F} = \mathbf{U}^\infty\cdot\mathbf{F}^* - \int_{S^P}\mathbf{u}\cdot\mathbf{f}^*\mathrm{d}S, \quad [19]$$

where * denotes the relevant quantities for the impermeable particle. By substituting for $\mathbf{u}$ in [19], we find the leading terms for $\mathbf{f}$ as a function of the surface force $\mathbf{f}^*$ on the impermeable particle.

$$\mathbf{U}^\infty\cdot\mathbf{F} = \mathbf{U}^\infty\cdot\mathbf{F}^* - \frac{\sqrt{k}}{\mu}\int_{S^P}(|\mathbf{f}^*|^2 - |\mathbf{f}^*\cdot\mathbf{n}|^2)\mathrm{d}S + O(k). \quad [20a]$$

It is interesting to note that [20a] can also be written in terms of the vorticity $\boldsymbol{\omega}^*$ on the surface of the impermeable particle:

$$\mathbf{U}^\infty\cdot\mathbf{F} = \mathbf{U}^\infty\cdot\mathbf{F}^* - \sqrt{k}\mu\int_{S^P}|\boldsymbol{\omega}^*|^2\mathrm{d}S + O(k). \quad [20b]$$

We note that this approach is easily generalized for a non-uniform velocity field $\mathbf{u}^\infty$. In that case, the reciprocal theorem is applied with the flow $\mathbf{u}^\infty$ past the porous particle, and a uniform flow past the impermeable particle.

For an impermeable sphere of radius a in a uniform flow, $\mathbf{f}^* = \frac{3}{2}\frac{\mu}{a}\mathbf{U}^\infty$ and hence from [20a], the force on a porous sphere is

$$\mathbf{F} = 6\pi\mu a\mathbf{U}^\infty\left[1 - \frac{\sqrt{k}}{a} + O\left(\frac{k}{a^2}\right)\right].$$

Figure 1 shows the variation of the force on the sphere as a function of permeability. The solid line is the result of the exact solution of Brinkman's equation as calculated by Felderhof (1975a). The dashed lines are the asymptotic results calculated here. It is seen that the asymptotic forms provide an excellent approximation over the entire range of permeability.

VI. EFFECTIVE VISCOSITY

It has been found that the bulk motion of a dilute suspension of freely suspended particles may be characterized by an effective viscosity (see Batchelor 1970). We proceed to determine

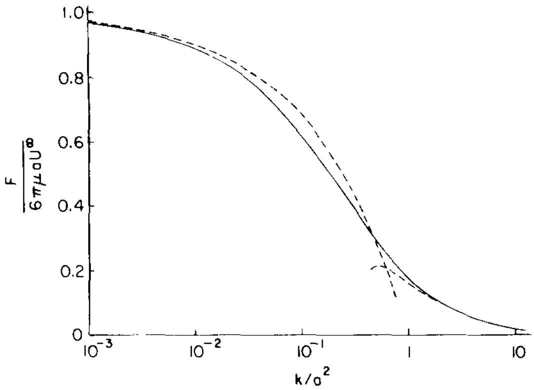

Figure 1. Hydrodynamic force on a porous sphere as a function of permeability: ————, exact solution; — — — —, asymptotic results.

this viscosity for a suspension of identical porous particles. Following Batchelor, we divide the bulk stress into two parts—the stress associated with the fluid and the additional stress due to the presence of the particles, where the particle stress is given by:

$$\sum_{ij} = \frac{\phi}{V^P}\int_{S^P} [\sigma_{ik}x_j n_k - \mu(u_i n_j + u_j n_i)]\mathrm{d}S, \tag{21}$$

and ϕ is the volume fraction occupied by the particles.

Applying the divergence theorem over a particle volume yields the alternative expression:

$$\sum_{ij} = \frac{\phi}{V^P}\int_{V^P} \frac{\mu}{k}[u_i - u_i^P]x_j \mathrm{d}V, \tag{22}$$

where the velocity $\mathbf{u}^P$ is now determined by the conditions of zero force and zero torque on the particle. In [21] and [22], the term contributing an isotropic stress has been neglected, because it has no effect on the motion.

To evaluate the effective viscosity, we must calculate the particle stress using either [21] or [22] in conjunction with the solution of the appropriate integral equation. For particles with large permeability, the leading terms are most easily found from [22], thus:

$$\sum_{ij} = \frac{\phi}{V^P}\left[\frac{\mu}{k}\int_{V^P}(u_i^\infty - u_i^P)x_j \mathrm{d}V - \frac{\mu^2}{k^2}\int_{V^P} x_j \int_{V^P} L_{0_{ik}}[u_k^\infty - u_k^P]\mathrm{d}V_y \mathrm{d}V_x\right]. \tag{23}$$

At this point, it is convenient to express the contribution of the particle stress in a nondimensional form. We define a viscosity coefficient β such that $\mu_{\text{eff}} = \mu(1+\beta\phi)$ where the effective viscosity μ_{eff} is determined by the particle stress.

For a suspension of porous spheres of radius a in a pure straining motion $e_{ij}x_j$, the particle stress is $\left(\frac{1}{5}\frac{a^2}{k} - \frac{2}{175}\frac{a^4}{k^2}\right)\mu\phi e_{ij}$ and the viscosity coefficient is $b = \frac{1}{10}\frac{a^2}{k} - \frac{1}{175}\frac{a^4}{k^2} + O\left(\frac{a^6}{k^3}\right)$.

To find the particle stress in the limit of small permeability, we employ the reciprocal theorem in a manner analogous to that used to calculate the hydrodynamic force. Consider two freely suspended particles of identical geometry, one impermeable, one porous, immersed in identical velocity fields $e_{ij}x_j$ where e_{ij} is a symmetric, traceless, constant tensor. Application of the reciprocal theorem together with [21] yields:

$$e_{ij}\sum_{ij} = e_{ij}\sum_{ij}^{*} - \frac{\phi}{V^P}\int_{S^P}(u_i - u_i^P)f_i^*\mathrm{d}S. \tag{24}$$

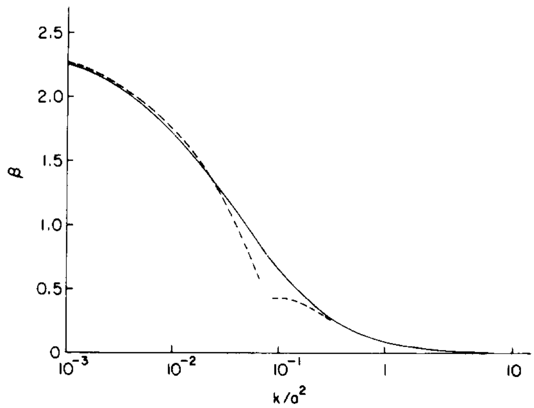

Figure 2. Viscosity coefficient as a function of permeability for a dilute suspension of porous spheres. ($\mu_{\text{eff}}/\mu = 1 + \beta\phi$): ———, exact solution; – – – – –, asymptotic results.

To first order this gives:

$$e_{ij}\sum_{ij} = e_{ij}\sum_{ij}^{*} - \frac{\phi}{V^P}\frac{\sqrt{k}}{\mu}\int_{S^P}(|\mathbf{f}^*|^2 - |\mathbf{f}^*\cdot\mathbf{n}|^2)\mathrm{d}S. \qquad [25]$$

For a suspension of spheres, radius a, substituting

$$f_i^* = \frac{5\mu}{a}e_{ij}x_j$$

into [25] gives:

$$\sum_{ij} = \mu\phi e_{ij}\left(5 - 15\frac{\sqrt{k}}{a}\right) \quad \text{and} \quad \beta = \frac{5}{2} - \frac{15}{2}\frac{\sqrt{k}}{a}.$$

Figure 2 shows the dependence of the viscosity coefficient β on permeability for suspensions of spheres. The solid line shows the result of Felderhof (1975 b) based on the exact solution of Brinkman's equation, while the dashed lines show the asymptotic forms. As before, we see that the asymptotic forms show excellent agreement with the exact result.

Acknowledgements—We would like to thank Prof. A. Acrivos for helpful comments and discussions. This work was supported in part by the National Science Foundation under grant ENG 78-09241. JJLH acknowledges the support of an NSF Postdoctoral Fellowship.

REFERENCES

Batchelor, G. K. 1970 The stress system in a suspension of force-free particles. *J. Fluid Mech.* **41**, 545.

Beavers, G. S. & Joseph, D. D. 1967 Boundary conditions at a naturally permeable wall. *J. Fluid Mech.* **30**, 197.

Brinkman, H. C. 1947 A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles, *Appl. Sci. Res.* **A1**, 27.

Debye, P. & Bueche, A.M. 1948 Intrinsic viscosity, diffusion, and sedimentation rate of polymers in solution. *J. Chem. Phys.* **16**, 573.

Felderhof, B. U. 1975a Frictional properties of dilute polymer solutions III. *Physica* **80A**, 63.

Felderhof, B. U. 1975b Frictional properties of dilute polymer solutions IV. *Physica* **80A**, 172.

Felderhof, B. U. & Deutch, J. M. 1975 Friction of dilute polymer solutions I. *J. Chem. Phys.* **62**, 2391.

FELDERHOF, B. U. & DEUTCH, J. M. 1975 Frictional properties of dilute polymer solutions II. *J. Chem. Phys.* **62**, 2398–2405.

HINCH, E. J. 1972 Note on the symmetries of certain material tensors for a particle in Stokes flow. *J. Fluid Mech.* **54**, 423.

HOWELLS, I. D. 1974 Drag due to the motion of a Newtonian fluid through a sparse random array of small fixed rigid objects. *J. Fluid Mech.* **64**, 449.

LADYZHENSKAYA, O. A. 1963 *The Mathematical Theory of Viscous Incompressive Flow.* Gordon & Beach, New York.

MATSUMOTO, K. & SUGANUMA, A. 1977 Settling velocity of a permeable model floc. *Chem. Engng Sci.* **32**, 445.

REULAND, P., FELDERHOF, B. U. & JONES, R. B. 1978 Hydrodynamic interaction of two spherically symmetric ploymers. *Physica* **93A**, 465.

SAFFMAN, P. G. 1971 On the boundary conditions at the surface of a porous medium. *Stud. Appl. Math.* **2**, 93.

YOUNGREN, G. K. & ACRIVOS, A. 1975 Stokes flow past a particle of arbitrary shape: a numerical method of solution. *J. Fluid Mech.* **69**, 377.

APPENDIX

Experimental verification of Brinkman's equation may be found in the work of Matsumoto & Suganuma (1977) who measured the settling velocity of model flocs made of steel wool. They calculated the settling velocity using Brinkman's equation and found excellent agreement with the experiments on spherical flocs with permeabilities ranging from $\sqrt{k}/a = 0.006$ to 0.3. In these experiments, the permeability was determined from independent measurements and was not used as an empirical parameter to match the data.

The experiments of Matsumoto and Suganuma show that Brinkman's equation accurately models the flow for one specific material; however, it would be premature to assume that it provides a reliable model for all types of permeable materials. To explore this point further, we consider the work of Beavers & Joseph (1967), who conducted experiments with rectilinear flow past blocks of porous material. As noted previously, these authors modelled the flow using Darcy's law with a slip condition and used the experimental data to determine the value of the slip constant α. For this simple rectilinear flow, Brinkman's equation with continuous boundary conditions is equivalent to Darcy's law with a slip constant $\alpha = 1$. For the experiments with the "foametal" material, this choice for α matches the data very nearly as well as the empirical values determined by Beavers & Joseph; however, for the "aloxite" material, Brinkman's equation shows poor agreement with the data.

The failure of Brinkman's equation to accurately describe the flow in the "aloxite" material may be explained by comparing the structure of this material with that of the "foametal". The "foametal" is composed of a lattice of metal fibres which create a set of irregualrly shaped interlocking pores with nearly uniform porosity throughout the material, whereas the "aloxite" is made from grains of crystalline aluminum oxide held together with a ceramic bond. Beavers & Jospeh hypothesized that this granular structure led to a higher porosity near the surface, and hence to a greater amount of slip in the boundary region. To accurately model this material with Brinkman's equation, we would have to specify the permeability as a function of position. As there is no obvious way to do this *a priori*, Brinkman's equation is no better than Darcy's law for materials of this type.

In conclusion, we find that we must exercise caution in using Brinkman's equation to model the flow in porous media. If the material is homogeneous from the surface of the body through the interior, Brinkman's equation will have a high chance of success; however, if the structure near the surface differs from that of the interior, modifications must be made in the form of variable permeability or discontinuous boundary conditions.